Journal of Analysis and Applications
Vol. 20 (2022), No.2, pp.91-104
ISSN: 0972-5954

URL : www.sasip.net

Further results of best simultaneous approximation on function spaces

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Abstract. The aim of this paper is to establish new results of best simultaneous proximinality problem for a finite number of vector valued functions in the Köthe spaces.

AMS Subject Classification (2020): 41A50, 41A28, 46B40

Keywords: Approximation, Köthe Bochner function spaces, Banach lattice

1. Introduction

The theory of best simultaneous approximation on operators and function spaces has been extensively investigated, see [1]-[9] [13]-[16]. Recent interests are focused on the study of the best simultaneous approximation in conditional complete Banach lattices paces with strong unit **1**, see [4], [6].

Definition 1. A lattice $(L, \leqslant)$ is said to be conditionally complete if it satisfies one of the following equivalent conditions:

(1) every non-empty lower bound set admits an infumum,

(2) every non-empty upper bound set admits a supremum,

(3) there exists a complete lattice $\bar{L} = L \cup \{\top, \bot\}$, which we shall call minimal completion of L, with bottom element $\bot$and top element $\top$, such that L is a sublattice of $\bar{L}$, $inf L = \bot$ and $sup L = \top$.

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Also recall that an element $\mathbf{1}$ of a Banach lattice space $(X, \|\cdot\|)$is called a strong unit if $\|\mathbf{1}\| = 1$and $x \leq \mathbf{1}$ for each element $x$in the unit ball of X.

Definition 2. A conditionally complete Banach lattice space X is a real Banach space which is also a conditionally complete vector lattice such that

$$|x| \leq |y| \Longrightarrow \|x\| \leq \|y\|,$$

for all $x, y \in X$, with $|x| = \max\{x, -x\}$.

We shall assume that X is a conditionally complete Banach lattice space with strong unit $\mathbf{1}$ throughout of this paper. The following Lemma can be found in [10], page 18:

Lemma 3. *Let* $(X, \|\cdot\|)$ *be a normed linear lattice. Then*

(1) $\| \ |x| \ \| = \|x\|$, for every x in X,

(2) if $x \wedge y = 0$, then $\|x - y\| = \|x + y\|$,

(3) the lattice operation are continuous,

(4) the positive cone of X is closed,

(5) the norm on X is regular and monotone.

Hence in this paper, one has$\| \ |x| \ \| = \|x\|$, for every x in X, and the norm in X is monontonic, that is

$$-y \leq x \leq y \Longrightarrow \|x\| \leq \|y\|,$$

for all $x, y \in X$.

Throughout this paper the measure space $(T, \sum, \mu)$ is a finite complete measure space.

Let $L^0 = L^0(T)$ be the space of all -equivalence classes-of Σ-measurable real-valued functions, where $x(t) \leq y(t)$ μ-almost every where ($a.e.t \in T$), if $x \leq y$ with $x,\ y \in L^0$.

Throughout this pape we'll use the symbol χ_A, where $A \in \sum$ to denote the characteristic function, which is specified to be one on A and zero elsewhere.

A Banach space $(E, \|\cdot\|_E)$ is termed a Köthe space, if for every x, $y \in L^0$, with $|x| \leq |y|$and $y \in E$, then x is an element of E and$\|x\|_E \leq \|y\|_E$, also if $\mu(A)$ is finite, where $A \in \Sigma$, then $\chi_A \in E$, see [11]. One can see that the space$E$ is a Banach lattice space under $\leq$.

If E is a Köthe space on the measure space $(T, \sum, \mu)$, then the space of all equivalence classes of strongly measurable functions $x : T \to X$, where $\|x(\cdot)\|$ is an element in E with the norm:

$$|||x||| = \| \|x(\cdot)\| \|_E \, .$$

Then $(E(X), |||\cdot|||_E)$ is a Banach spaces called the Köthe Bochner function space induced by E and X. Also if X is Banach lattice, $E(X)$ is also a Banach lattice space, see [11].

Let G be a closed subspace of X, which is a Banach space. Let us define the norm on $X^n = X \times X \times \ldots \times X$ on a finite number of elements $x_1, \ldots x_n$ in X, by:

$$\|(x_1, x_2, \ldots, x_n)\|_m = \max\{\|x_1\|, \ldots, \|x_n\|\},$$

where X^n with the norm $\|\cdot\|_m$defined above is denoted by $\underset{m}{\otimes} X^n$.

Set $W = \{(y, \ldots, y) \text{ n-tuple} : y \in G\}$, define the norm on W by

$$\|(y, \ldots, y)\|_m = \|y\|, \; y \in G$$

then $\underset{m}{\otimes} X^n$ with the norm $\|\cdot\|_m$ is a Banach lattice space with W a closed subspace of $\underset{m}{\otimes} X^n$. If for any finite elements $x_1, \ldots, x_n$ in X, and there exists an element y_o in G, such that

$$\begin{aligned} d(x_1, \ldots, x_n, G) &= \inf_{y \in G} \max\{\|x_1 - y\|, \ldots, \|x_n - y\|\} \\ &= \max\{\|x_1 - y_0\|, \ldots, \|x_n - y_0\|\}, \end{aligned} \tag{1}$$

then we call the element y_o a best $max-$simultaneous proximinant $(BMSP)$ of the finite elements $x_1, \ldots, x_2$ in X and G a $max-$simltaneous proximinal (MSP) in X.

We state that G is $max-$simultaneously Chebyshev in X , if such y_o in G is a unique element. Note that G is (MSP) in X if and only if W is proximinal in $\underset{m}{\otimes} X^n$.

Remark 4. The best $max-$simultaneous proximinant $(BMSP)$ appears to be stronger than ordinary proximinality for $x \in X$, and $d(x, G) = \inf_{y \in G} \|x - y\|$. However as cited in [13], this is not the case in general for various characterizations of simultaneous approximations.

The norm on f is defined by

$$|||f|||_m = \|\max\{\|x_1(\cdot)\| \ , \ldots, \|x_n(\cdot)\|\}\|_E \,,$$

for a function $f = (x_i)_{i=1}^n \in (E(X))^n$.

We show the existence of the element $y_0 = (g_0, \ldots, g_0)$ in $(E(G))^n$ for a given closed subspace G of X and $f = (x_i)_{i=1}^n$ in $(E(X))^n$, where the infimum obtained

$$\begin{aligned} |||f - y_0|||_m &= \inf_{g \in E(G)} |||f - (g, \ldots, g)|||_m \\ &= \|\max\{\|x_1(\cdot) - g_0(\cdot)\| \ , \ldots, \|x_n(\cdot) - g_0(\cdot)\|\}\|_E \ , \end{aligned}$$

which means that y_0 is a $(BMSP)$ of $f = (x_i)_{i=1}^n$ in $(E(X))^n$.

Then

$$d(x_1,\ldots,x_n,E(G)) = \inf_{g\in E(G)} \|\max\{\|x_1(\cdot)-g(\cdot)\| \ ,\ldots,\|x_n(\cdot)-g(\cdot)\|\}\|_E$$
$$= \|\max\{\|x_1(\cdot)-g_0(\cdot)\| \ ,\ldots,\|x_n(\cdot)-g_0(\cdot)\|\}\|_E$$

In this paper, we We investigate the$(BMSP)$ for finite elements in $E(X)$.

2. Distance formula

For $x_1,\ldots,x_n$ in $E(X)$, we define the set $Best(x_1,\ldots,x_n,E(G))$ by

$$\left\{ \begin{array}{c} g\in E(G): \|\max\{\|x_1(\cdot)-g(\cdot)\| \ ,\ldots,\|x_n(\cdot)-g(\cdot)\|\}\|_E \\ = d(x_1,\ldots,x_n,E(G)) \end{array} \right\}.$$

It is clear that if the set $Best(x_1,\ldots,x_n,E(G))$ is non-empty, then $E(G)$ is (MSP) in $E(X)$.

Lemma 5. *If $x_1,\ldots,x_n$ are elements in $E(X)$, and a strongly measurable function $g: T\to G$, with $g(t)\in Best(x_1(t),\ldots,x_n(t),G)$ for a. e.$t\in T$, then g is an element of $E(G)$ and also $g\in Best(x_1,\ldots,x_n,E(G))$.*

Proof. Since $g(t)$ belong to the set $Best(x_1(t),\ldots,x_n(t),G)$, (a. e.$t\in T$), we have

$$\begin{aligned}\|g(t)\| &\le \|x_1(t)-g(t)\| \ + \|x_1(t)\| \\ &\le \max\{\|x_1(t)-g(t)\|,\ldots,\|x_n(t)-g(t)\|\} + \|x_1(t)\| \\ &\le \max\{\|x_1(t)\| \ ,\ldots,\|x_n(t)\|\} + \|x_1(t)\| \\ &\le \|x_1(t)\| \ + \sum_{i=1}^{n}\|x_i(t)\|.\end{aligned}$$

Hence

$$|\|g\|| \le |\|x_1\|| + \sum_{i=1}^{n} |\|x_i\||.$$

This means that g is an element in $E(G)$. But for all $h \in E(G)$, we have

$$\max\{\|x_1(t)-g(t)\| \ ,\ldots,\|x_n(t)-g(t)\|\}$$
$$\leq \max\{\|x_1(t)-h(t)\| \ ,\ldots,\|x_n(t)-h(t)\|\},$$

Hence

$$\|\max\{\|x_1(\cdot)-g(\cdot)\| \ ,\ldots,\|x_n(\cdot)-g(\cdot)\|\}\|_E$$
$$\leq \|\max\{\|x_1(\cdot)-h(\cdot)\| \ ,\ldots,\|x_n(\cdot)-h(\cdot)\|\}\|_E \ .$$

Thus g is an element of $Best(x_1,\ldots,x_n,E(G))$. □

Definition 6 [11]. If for every element $x$in E and for every decreasing sequence $(A_k)_{k=1}^{\infty} \subseteq E$, converging to 0, we have $\lim_{k\to\infty} \|\chi_{A_k}\ x\|_E = 0$, then the norm $\|\cdot\|_E$is called an absolute continuous norm on E .

Definition 7 [12]. If for $x \leq y$ and $\|x\|_E = \|y\|_E$ implies that $x = y$, then E is said to be strictly monotone.

Now, we introduce the following theorem of the distance equality:

Theorem 8. *Assume that $|||\cdot|||$ is an absolute continuous norm on $E(X)$. If $x_1,\ldots,x_n$ are finite number of functions in $E(X)$, then the function $d(x_1,\ldots,x_n,E(G))$ is an element of E and*

$$\|d(x_1(\cdot),\ldots,x_n(\cdot),G)\|_E = d(x_1,\ldots,x_n,E(G)).$$

Proof. If $x_1,\ldots,x_n$ in $E(X)$, then there exists a finite number of sequence of simple functions in $E(X)$, say $(x_{k,i})_{k=1}^{\infty}$,($i=1,\ldots,n$), such that for any $i,((1\leq i\leq n))$, we obtain

$$\lim_{k\to\infty}\|x_{k,i}(t)-x_i(t)\| = 0, (a.e.\ t\in T).$$

Since the function $d(x_1(\cdot),\ldots,x_n(\cdot),G)$ is continuous, then for each i, $(1\leq i\leq n)$ and $t\in T$, there exists a sequence of elements $(x_{k,i}(t))_{k=1}^{\infty}\subseteq G$

such that

$$\lim_{k\to\infty} \left| d\left(x_{k,1}\left(t\right),,\ldots,x_{k,n}\left(t\right),G\right) - d\left(x_1\left(t\right),,\ldots,\ x_n\left(t\right),G\right)\right| = 0,$$

Now, for each $k \in$ and $t \in T$, set

$$M_k\left(t\right) = d\left(x_{k,1}\left(t\right),,\ldots,x_{k,n}\left(t\right),G\right),$$

then each $k = 1, 2, \ldots, \infty$, M_k is a measurable function.

Therefore, $d\left(x_1\left(\cdot\right),\ldots,x_n\left(\cdot\right),G\right)$ is measurable, also for all $z \in G$, and $t \in T$, we have

$$d\left(x_1\left(t\right),\ldots,\ x_n\left(t\right),G\right) \leq \max\left\{\left\|x_1\left(t\right) - z\right\|\ ,\ldots,\left\|x_n\left(t\right) - z\right\|\right\},$$

Also, for all $g \in E\left(G\right)$, we get

$$d\left(x_1\left(t\right),\ldots,\ x_n\left(t\right),G\right) \leq \max\left\{\left\|x_1\left(t\right) - g\left(t\right)\right\|\ ,\ldots,\left\|x_n\left(t\right) - g\left(t\right)\right\|\right\},$$

Then

$$\begin{aligned} &\left\|d\left(x_1\left(\cdot\right),\ldots,\ x_n\left(\cdot\right),G\right)\right\|_E \\ &\leq \left\|\max\left\{\left\|x_1\left(\cdot\right) - g\left(\cdot\right)\right\|\ ,\ldots,\left\|x_n\left(\cdot\right) - g\left(\cdot\right)\right\|\right\}\ \right\|_E\ . \end{aligned}$$

This implies that $d\left(x_1\left(\cdot\right),\ldots,\ x_n\left(\cdot\right),G\right)$ is an element of E and

$$\left\|d\left(x_1\left(\cdot\right),\ldots,\ x_n\left(\cdot\right),G\right)\right\|_E \leq d\left(x_1,\ldots,x_n,E\left(G\right)\right). \tag{2}$$

The simple functions are dense in $E(X)$ because $|||\cdot|||$ is an absolute continuous norm on $E\left(X\right)$, [11]. Thus in $E\left(X\right)$ there are simple functions x_i^* such that for $\varepsilon > 0$, we have

$$|||x_i^* - x_i|||\ \ < \frac{\varepsilon}{n},\ \ (for\ i = 1,\ldots,n).$$

Let us suppose that

$$x_i^*(t) = \sum_{k=1}^{m} \alpha_k^i \chi_{A_k}(t), (for\ i = 1,\ldots,n).$$

where A_k's are measurable sets with $A_i \cap A_j = \varphi$ for $i \neq j$, also we assume that $\mu(A_k) > 0$, for each $k = 1, 2, ..., m$ and $T = \bigcup_{k=1}^{\infty} A_k$, where $\alpha_k^i \in X$, $k = 1, 2, ..., m$, and for $i = 1, \dots, n$.

Let $c = |||\chi_T||| > 0$, then for each $k = 1, 2, \cdots, m$, and $y_k \in G$ satisfying

$$\max\left\{\left\|\alpha_k^1 - y_k\right\|, \dots, \left\|\alpha_k^n - y_k\right\|\right\} \leq d\left(\alpha_k^1, \dots, \alpha_k^n, G\right) + \frac{\varepsilon}{c}.$$

Consider

$$g(t) = \sum_{k=1}^{m} y_k \ \chi_{A_k}(t), \ \ for \ t \in T$$

Thus, we can obtain the following inequality

$$\begin{aligned}
&\max\left\{\left\|\alpha_1^*(t) - g(t)\right\|, \dots, \left\|\alpha_n^*(t) - g(t)\right\|\right\} \\
&= \sum_{k=1}^{m} \chi_{A_k}(t) \max\left\{\left\|\alpha_k^1 - y_k\right\|, \dots, \left\|\alpha_k^n - y_k\right\|\right\} \\
&\leq \sum_{k=1}^{m} \chi_{A_k}(t) \left[d\left(\alpha_k^1, \dots, \alpha_k^n, G\right) + \frac{\varepsilon}{c}\right] \\
&= d\left(x_1^*(t), \dots, , x_n^*(t), G\right) + \frac{\varepsilon}{c} \sum_{k=1}^{m} \chi_{A_k}(t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|\max\left\{\left\|x_1^*(\cdot) - g(\cdot)\right\|, \dots, , \left\|x_n^*(\cdot) - g(\cdot)\right\|\right\}\right\|_E \\
&\leq \left\|d\left(x_1^*(\cdot), \dots, x_n^*(\cdot), G\right)\right\|_E + \frac{\varepsilon}{c} \left|\left|\left|\sum_{k=1}^{m} \chi_{A_k}\right|\right|\right| \\
&\leq \left\|d\left(x_1^*(\cdot), \dots, x_n^*(\cdot), G\right)\right\|_E + \frac{\varepsilon}{c} |||\chi_T||| \\
&= \left\|d\left(x_1^*(\cdot), \dots, x_n^*(\cdot), G\right)\right\|_E + \varepsilon.
\end{aligned}$$

It follows that

$$\begin{aligned} d\left(x_1,\ldots,x_n,E\left(G\right)\right) &\le d\left(x_1^*,\ldots,x_n^*,E\left(G\right)\right)+\sum_{i=1}^{n}\left|\left\|x_i-x_i^*\right\|\right| \\ &< \left\|\max\left\{\left\|x_1^*\left(\cdot\right)-g\left(\cdot\right)\right\|,\ldots,\left\|x_n^*\left(\cdot\right)-g\left(\cdot\right)\right\|\right\}\right\|_E+\varepsilon \\ &\le \left\|d\left(x_1^*\left(\cdot\right),\ldots,x_n^*\left(\cdot\right),G\right)\right\|_E+2\varepsilon \\ &\le \left\|d\left(x_1\left(\cdot\right),\ldots,x_n\left(\cdot\right),G\right)\right\|_E+\sum_{i=1}^{n}\left|\left\|x_i-x_i^*\right\|\right|+2\varepsilon \\ &< \left\|d\left(x_1\left(\cdot\right),\ldots,x_n\left(\cdot\right),G\right)\right\|_E+3\varepsilon. \end{aligned}$$

Then

$$d\left(x_1,\ldots,x_n,E\left(G\right)\right)<\left\|d\left(x_1\left(\cdot\right),\ldots,x_n\left(\cdot\right),G\right)\right\|_E+3\varepsilon.$$

It holds that

$$d\left(x_1,\ldots,x_n,E\left(G\right)\right)\le\left\|d\left(x_1\left(\cdot\right),\ldots,x_n\left(\cdot\right),G\right)\right\|_E. \tag{3}$$

As a result, the inequalities (2) and (3) produce the desired result. □

The following is the consequence of the preceding theorem:

Corollary 9. *If the norm* $|\|\cdot\||$ *on* $E\left(X\right)$ *is absolute continuous and strictly monotone, then for a finite number of functions* $x_1,\ldots,x_n$ *in* $E\left(X\right)$*,* g *is an element of* $Best\left(x_1,\ldots,x_n,E\left(G\right)\right)$ *if and only if*

$$g\left(t\right)\in Best\left(x_1\left(t\right),\cdots,x_n\left(t\right),G\right),\left(a.e.\ t\in T\right).$$

The following result focuses on the characterization of the (MSP) of simple functions in $E(X)$:

Theorem 10. *If* G *is*(MSP) *in* X*, then for every finite elements of simple functions* $x_1,\ldots,x_n$ *in* $E(X)$*, the set* $Best\left(x_1,\ldots,x_n,E\left(G\right)\right)$ *is a non-empty set.*

Proof. Let $x_1,\ldots,x_n$ be a finite number of simple functions in $E(X)$. Each of these functions can be written as

$$x_i(t)=\sum_{k=1}^{m}\alpha_k^i\ \chi_{A_k}(t), i=1,\ldots n.$$

$i=1,\ldots,n$,where $T=\bigcup_{k=1}^{\infty}A_k$, where A_k's are measurable sets and $(A_i\cap A_j=\varphi$ for $i\neq j)$, we also suppose that $\mu(A_k)>0$, for each $k=1,2,...,m$.

Then, we know that for each $k=1,2,\cdots,m$, there exists $(BMSP)$ elements w_k in G of the finite number of elements $\left(\alpha_k^1\ ,\ldots,\alpha_k^n\ \right)$ in $\underset{m}{\otimes}X^n$ such that

$$d\left(x_k^1,\ldots,x_k^n\ ,G\right)=\max\left\{\left\|\alpha_k^1-z_k\right\|,\ldots,\left\|\alpha_k^n-z_k\right\|\right\}.$$

Set

$$g(t)=\sum_{k=1}^{m}z_k\ \chi_{A_k}(t),(t\in T),$$

then for any function h in $E(G)$ and any $\alpha>0$, we get

$$\begin{aligned}&\left\|\max\left\{\left\|x_1(\cdot)-h(\cdot)\right\|,\ldots,\left\|x_n(\cdot)-h(\cdot)\right\|\right\}\right\|_E\\ &\geq\left\|\sum_{k=1}^{m}\chi_{A_k}(\cdot)\left[\max\left\{\left\|\alpha_k^1-z_k\right\|,\ldots,\left\|\alpha_k^n-z_k\right\|\right\}\right]\right\|_E\\ &=\left\|\max\left\{\left\|x_1(\cdot)-g(\cdot)\right\|,\ldots,\left\|x_n(\cdot)-g(\cdot)\right\|\right\}\right\|_E.\end{aligned}$$

Taking infimum over all functions h, we get

$$d\left(x_1,\ldots,x_n,E(G)\right)=\left\|\max\left\{\left\|x_1(\cdot)-g(\cdot)\right\|,\ldots,\left\|x_n(\cdot)-g(\cdot)\right\|\right\}\ \right\|_E.$$

As a result, the finite elements of simple functions $x_1,\ldots,x_n$ are $(BMSP)$ in $E(X)$. □

Theorem 11. *Let the norm* $|||\cdot|||$ *be an absolute continuous and strictly monotone norm on* $E(X)$*. If* $E(G)$ *is* (MSP) *in* $E(X)$*, then* G *is* (MSP) *in* X*.*

Proof. Let $\alpha_1, \ldots, \alpha_n \in X$. Set $x_i(t) = \alpha_i \quad (i = 1, \ldots, n) \quad (a.e.\ t \in T)$. Since

$$\begin{aligned} |x_i| &= \| \|x_i(\cdot)\| \ \|_E = \| \|\alpha_i \chi_T(\cdot)\| \|_E \\ &= \|\alpha_i\| \quad \| |\chi_T| \|, \quad (i = 1, \ldots, n). \end{aligned}$$

which is finite, then $x_i \in E(X)$, for each $i, (i = 1, \ldots, n)$.

By assumption, there exists a function g in $E(G)$ that satisfies the following

$$\begin{aligned} &\|\max\{\|x_1(\cdot) - g(\cdot)\|, \ldots, \|x_n(\cdot) - g(\cdot)\|\} \ \|_E \\ &\leq \|\max\{\|x_1(\cdot) - h(\cdot)\|, \ldots, \|x_n(\cdot) - h(\cdot)\|\}\|_E, \end{aligned}$$

for all $h \in E(G)$. Since $E(X)$ is a Köthe Bochner function space with a strictly monotone norm, then for almost $t \in T$, we have

$$\begin{aligned} &\max\{\|x_1(t) - g(t)\|, \ldots, \|x_n(t) - g(t)\|\} \\ &\leq \max\{\|x_1(t) - h(t)\|, \ldots, \|x_n(t) - h(t)\|\}. \end{aligned}$$

Fix $t_0 \in T$ and $y = g(t_0)$, then $y \in G$ and for all $h \in E(G)$, we have

$$\max\{\|\alpha_1 - y\|, \ldots, \|\alpha_n - y\|\} \leq \max\{\|\alpha_1 - h(t)\|, \ldots, \|\alpha_n - h(t)\|\},$$

Since G is embedded isometrically into $E(G)$, it follows that

$$\max\{\|\alpha_1 - y\|, \ldots, \|\alpha_n - y\|\} \leq \max\{\|\alpha_1 - w\|, \ldots, \|\alpha_n - w\|_X\},$$

for all $w \in G$. □

3. Conclusion

The best simultaneous approximations of a finite number of functions in Köthe Bochner function spaces in the maximal sense were studied in this paper. The relationship between the $(BMSP)$ of G, the closed subspace of

X and the $(BMSP)$ of $E(G)$ in $E(X)$ were also addressed. These characterization can be viewed as an extension of a number of related theorems about Orlicz Bochner spaces and Lp Bochner spaces.

Acknowledgements. The authors would like to thank the anonymous reviewers for their careful reading of their manuscript and their many insightful comments and suggestions.

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(*Received: September, 2021; Revised: October, 2021*)